

Symmetric Functions and Representations of Quantum Affine Algebras

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ABSTRACT. We study connections between the ring of symmetric functions and the characters of irreducible finite-dimensional representations of quantum affine algebras. We study two families of representations of the symplectic and orthogonal Lie algebras. One is defined via combinatorial properties and is easy to calculate; the other is closely related to the $q = 1$ limit of the “minimal affinization” representations of quantum affine algebras. We conjecture that the two families are identical, and present supporting evidence and examples. In the special case of a highest weight that is a multiple of a fundamental weight, this reduces to a conjecture of Kirillov and Reshetikhin, recently proved by the first author.

0. Introduction

In this paper we study connections between the ring of symmetric functions and the characters of irreducible finite-dimensional representations of quantum affine algebras. We introduce the reader to two families of representations of the classical finite-dimensional simple Lie algebras \mathfrak{g} , indexed by dominant integral weights λ . One family is defined via combinatorics and the ring of symmetric functions, and is easy to describe and calculate. The other family consists of “minimal affinizations” [1], certain representations of quantum affine algebras, regarded as representations of the underlying finite-dimensional algebra. We conjecture that these two families are identical and prove the conjecture in certain cases. In addition, we establish a number of results which provide compelling evidence for the conjecture and also illuminate the structure of the minimal affinizations of quantum groups.

In Section 1 we define the representations $W_{Sp}(\lambda)$ and $W_O(\lambda)$ of the symplectic and orthogonal algebras, respectively. They are described in terms of their universal characters, which are elements of the ring of symmetric functions. They have the remarkable property that the map taking the Schur function s_λ to the character of $W_G(\lambda)$ is an isomorphism of the ring of symmetric functions ($G = Sp$ or O). This condition suffices to define the representations $W_G(\lambda)$ completely. In the special case when λ is a multiple of a fundamental weight (also called a rectangle,

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from the shape of its Young diagram), the modules $W_G(\lambda)$ were defined earlier by Kirillov and Reshetikhin ([8], see also [6]); the fact that this assignment extends to a homomorphism of rings was proved in [11].

In Section 2 we define representations $W_{\mathfrak{g}}^{\text{aff}}(\lambda)$ of the loop algebra $L(\mathfrak{g})$. They have as quotients the $q = 1$ specialization of the “minimal affinization,” a canonical representation of $U_q(\hat{\mathfrak{g}})$ associated to each λ . In [3] it was proved that when λ is a rectangle, they are in fact isomorphic and that

$$W_{\mathfrak{g}}^{\text{aff}}(\lambda) \cong W_G(\lambda),$$

a result that was conjectured in [8]. In this paper, we conjecture that this is true for all λ . Most of the section is devoted to proving results on the \mathfrak{g} -module structure of the modules $W_{\mathfrak{g}}^{\text{aff}}(\lambda)$ which support this conjecture. We isolate crucial properties that are known to be true for one family and prove that the other family also satisfies them.

1. The modules $W_G(\lambda)$ are defined for $so(n)$ and $sp(2n)$ for all n . Central to their definition is the fact that the direct sum decomposition into irreducibles is independent of n , provided n is sufficiently large. We show that the $W_{\mathfrak{g}}^{\text{aff}}(\lambda)$ share this property.
2. The minimal affinization is distinguished from all other affinizations of λ by the property that if it contains a \mathfrak{g} -highest weight vector with weight μ , then the root $\lambda - \mu$ cannot be contained in a sub-root-lattice for a subalgebra isomorphic to some $sl(r)$. We show that $W_G(\lambda)$ and $W_{\mathfrak{g}}^{\text{aff}}(\lambda)$ both have this property.
3. Finally, in Section 3.3 we calculate examples. We restrict our attention to $\mathfrak{g} = so(2n)$ for convenience; the proofs in the other cases are similar. For several types of weights λ , we completely calculate $W_O(\lambda)$ and show explicitly that $W_{\mathfrak{g}}^{\text{aff}}(\lambda)$ is a submodule.

We also mention other properties which we can prove for one of $W_G(\lambda)$ or $W_{\mathfrak{g}}^{\text{aff}}(\lambda)$ and which the other seems empirically to share, though we cannot as yet provide a proof.

The structure of finite-dimensional representations of quantum affine algebras is very complicated. Establishing the conjecture in this paper would significantly expand our understanding of their algebraic and combinatorial structure. The work of [8] also conjectured a formula for decomposing tensor products of representations associated to rectangles, given by the fermionic formula. A generalization of that formula beyond rectangles using the representations studied here would be of considerable interest.

1. Symmetric Functions

In this section we fix notation and recall the basic notions of the ring Λ of symmetric functions as a tool for handling representations of the classical Lie algebras. Our goal is the definition of a certain subcategory of finite-dimensional representations of the orthogonal and symplectic Lie algebras. This subcategory is closed under taking direct sums and tensor products, and it is generated as an abelian group by a family of modules $W(\lambda)$, as λ runs over all dominant integral highest weights. It has the remarkable property that the multiplicities in the decomposition of tensor products are the Littlewood–Richardson numbers.

1.1. Some classical bases. We will work in the ring Λ of formal symmetric functions in countably many variables (x_1, x_2, \dots) , and primarily follow the notation of Macdonald [13], to which we refer the reader for proofs of fundamental facts. Our emphasis is the dictionary which translates between the combinatorics of symmetric functions and the representation theory of the Lie algebra \mathfrak{sl}_n .

The k th *complete* symmetric function h_k is the sum of all monomials of degree k in the variables (x_1, x_2, \dots) . The k th *elementary* symmetric function e_k is the sum of all square-free monomials of degree k in the variables (x_1, x_2, \dots) . These functions are clearly symmetric, *i.e.* invariant under all permutations of the x_i . Each of these sets is algebraically independent, and Λ is exactly the ring of polynomials in either the h 's or the e 's.

We can specialize these functions to polynomials by setting all variables except for x_1, \dots, x_n to be zero, for any positive integer n . Now they are intimately familiar to representation theorists: note that $h_1 = e_1$ is the character of the fundamental n -dimensional vector representation V of the Lie group $GL(n)$ or the Lie algebra \mathfrak{sl}_{n+1} . More generally, h_k is the character of $S^k(V)$, the k th symmetric power of the vector representation, while e_k is the character of $\Lambda^k(V)$, the k th alternating power; all of these are irreducible representations.

More precisely, recall that \mathfrak{sl}_{n+1} has n *fundamental weights* $\omega_1, \dots, \omega_n$, and that its finite-dimensional irreducible representations are indexed by *dominant* weights, positive integer linear combinations of the fundamental weights. We will write $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ for such a weight and $V(\lambda)$ for the associated representation; λ is called the *highest weight*. Then h_k specializes to the character of $V(k\omega_1)$, while e_k specializes to the character of $V(\omega_k)$.

The *Schur functions* fill out this picture. For each dominant weight λ , the Schur function $s_\lambda \in \Lambda$ is a symmetric function in (x_1, x_2, \dots) , and when we specialize to a polynomial by setting $x_i = 0$ for $i > n$, we get the character of the representation $V(\lambda)$ of \mathfrak{sl}_{n+1} . Note that n must be large enough that λ involves only the first n fundamental weights. It is more traditional to index the Schur functions s_λ by *partitions*, integer sequences $\lambda = \langle \lambda_1, \dots, \lambda_r \rangle$ with $\lambda_1 \geq \dots \geq \lambda_r \geq 0$, or by their graphical representation *Young diagrams*, in which λ is depicted as an array of r rows of boxes, left-justified, with λ_i boxes in the i th row. Translation between a weight and a partition is straightforward: the coefficient in λ of ω_k is $\lambda_k - \lambda_{k+1}$, or the number of columns of height exactly k in the Young diagram. For example:

$$\lambda = \omega_1 + 2\omega_2 + \omega_3, \quad \lambda = \langle 4, 3, 1 \rangle, \quad \text{Young diagram} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$$

Thus the finite-dimensional irreducible representations of \mathfrak{sl}_{n+1} are indexed by the Young diagrams with at most n rows.

Since the ring of symmetric functions is the ring of polynomials in the h 's, we must be able to write s_λ as such a polynomial. This is accomplished by the *Jacobi-Trudi* identity:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1,\dots,r}, \quad \text{where } \lambda = \langle \lambda_1, \dots, \lambda_r \rangle.$$

We can use this to write s_λ in terms of the e 's as well once we introduce the involutive ring automorphism $\omega : \Lambda \rightarrow \Lambda$, defined by any one of the following:

$$\omega(h_k) = e_k, \quad \omega(e_k) = h_k, \quad \omega(s_\lambda) = s_{\lambda'},$$

where λ' is the *conjugate* of λ , whose Young diagram is obtained from that of λ by reflecting it through its main diagonal, exchanging rows and columns. Note that

$h_k = s_{\langle k \rangle}$, whose Young diagram is a single row of length k , and $e_k = s_{\langle 1, \dots, 1 \rangle}$, whose Young diagram is a single column of height k .

The Schur functions s_λ form a linear basis for Λ as λ runs through all partitions (including the empty partition; $s_\emptyset = 1$ corresponds to the trivial representation, with highest weight 0). We endow Λ with an inner product $\langle \cdot, \cdot \rangle$ by declaring this basis orthonormal. The multiplication in Λ has as its structure constants the *Littlewood–Richardson numbers*:

$$(1.1) \quad c_{\mu\nu}^\lambda := \langle s_\lambda, s_\mu s_\nu \rangle, \quad \text{or equivalently, } s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda.$$

We write $|\lambda|$ for the number of boxes in the Young diagram of λ , which is also the degree of the monomials in s_λ . As a result,

$$(1.2) \quad c_{\mu\nu}^\lambda = 0 \quad \text{unless} \quad |\lambda| = |\mu| + |\nu|.$$

Since Schur functions are characters, the Littlewood–Richardson numbers describe the decomposition of a tensor product of representations of \mathfrak{sl}_n into irreducibles: $c_{\mu\nu}^\lambda$ is the multiplicity of $V(\lambda)$ in $V(\mu) \otimes V(\nu)$. Thus the Littlewood–Richardson numbers are nonnegative integers.

1.2. Symplectic and orthogonal analogues. We now turn to the other classical Lie algebras, those of the symplectic and orthogonal (of even or odd dimension) groups. The work of Koike and Terada [9] showed that they too have “universal characters” in Λ , which specialize to characters of their representations in a way analogous to the Schur functions and \mathfrak{sl}_n .

The irreducible finite-dimensional representations of the Lie groups $SO(2n+1)$ and $Sp(2n)$ are once again indexed by highest weights which are, as above, in bijection with Young diagrams with at most n rows. The irreducible representations of $O(2n)$ are indexed by the same set, but when we restrict to $SO(2n)$, representations associated to Young diagrams with exactly n rows split into two irreducibles (exchanged by the automorphism switching the two “spin” weights). The starting point of [9] is the observation that the characters of these representations, just as in the case of $GL(n)$, are “stable,” in the sense that they are all specializations of their $n \rightarrow \infty$ limit.

These “stable limit” or “universal” characters form two new bases for Λ . One consists of the characters sp_λ coming from the symplectic groups, and the other of the characters o_λ coming from the orthogonal groups (which give one stable limit, independent of the parity of their rank). We can specialize o_λ and sp_λ to get characters of irreducible orthogonal or symplectic representations of $SO(2n+1)$, $SO(2n)$ or $Sp(2n)$ as long as n is large enough; for our purposes we note that n is sufficiently large when the number of nonzero parts of λ is at most $n-1$ (or $n-2$ for $SO(2n)$). When n is too small we get a character of a reducible representation; we refer readers to the original paper or a well-written summary, like Appendix A of [5], for this level of details of the specialization homomorphisms.

These two new bases have their own structure constants, the symplectic and orthogonal analogues of the Littlewood–Richardson numbers. The following remarkable fact deserves wider recognition.

THEOREM 1.1. *There is a collection of nonnegative integers $d_{\mu\nu}^\lambda$ such that*

$$sp_\mu sp_\nu = \sum_\lambda d_{\mu\nu}^\lambda sp_\lambda \quad \text{and} \quad o_\mu o_\nu = \sum_\lambda d_{\mu\nu}^\lambda o_\lambda.$$

That is, the structure constants of the symplectic and orthogonal bases are the same.

The equality is shown in [9] in terms of symmetric functions; it also follows easily from crystal base theory. The $d_{\mu\nu}^\lambda$ are certainly nonnegative integers, as they count the multiplicity of $V_G(\lambda)$ in $V_G(\mu) \otimes V_G(\nu)$. Here the V_G can denote representations of any one of $G = O(2n+1)$, $Sp(2n)$, or $O(2n)$, so long as we require n to be sufficiently large — in particular, larger than the sum of the numbers of rows in μ and ν . For example,

$$\begin{aligned} V(\omega_1)^{\otimes 2} &\cong V(2\omega_1) \oplus V(\omega_2) && \text{in } GL(n) \text{ or } SL(n+1), n \geq 2, \\ V_G(\omega_1)^{\otimes 2} &\cong V_G(2\omega_1) \oplus V_G(\omega_2) \oplus \mathbf{C} && \text{in } SO(2n+1), Sp(2n), \text{ or } SO(2n), n > 2. \end{aligned}$$

Here \mathbf{C} denotes the trivial representation $V_G(0)$.

In some sense the $d_{\mu\nu}^\lambda$ are a deformation of the Littlewood–Richardson numbers. Analogous to (1.2), we have

$$(1.3) \quad d_{\mu\nu}^\lambda = 0 \quad \text{unless} \quad |\lambda| = |\mu| + |\nu| - 2k, \quad k \in \mathbf{Z}_{\geq 0}.$$

Recall that for $c_{\mu\nu}^\lambda$ we demand this with $k = 0$. Moreover,

$$(1.4) \quad c_{\mu\nu}^\lambda = d_{\mu\nu}^\lambda \quad \text{when} \quad |\lambda| = |\mu| + |\nu|.$$

In other words, moving from the general linear to the symplectic or orthogonal groups only adds new pieces to the decomposition of tensor products, and all the new pieces are lower-order terms.

From the representation theory point of view, λ is a weight and can be written as a linear combination of the fundamental roots $\alpha_1, \dots, \alpha_n$. As long as $k < n$ (or $n-1$, for type D_n), the coefficient of α_k is the number of boxes in the top k rows of the Young diagram of λ , so $|\lambda|$ is the coefficient of α_k for k greater than the number of rows of λ . (The coefficient of α_n may differ from $|\lambda|$ but only by a factor of two.) Therefore the “extra pieces” of $V(\mu) \otimes V(\nu)$ corresponding to nonzero $d_{\mu\nu}^\lambda$ with $|\lambda| < |\mu| + |\nu|$ can be identified by the fact that the weight $\mu + \nu - \lambda$ is supported on the “spin” or “long” root α_n (and for $SO(2n)$ on α_{n-1} also).

1.3. A new family of representations. We are now ready to define the symplectic and orthogonal families of representations $W(\lambda)$. The stable limit characters of the W ’s will form another pair of new bases of the ring Λ , and the representations are completely characterized by the property that the structure constants of these new bases are the classical Littlewood–Richardson numbers $c_{\mu\nu}^\lambda$.

Consider the natural inclusions $SO(2n+1) \subseteq GL(2n+1)$, $Sp(2n) \subseteq GL(2n)$, and $SO(2n) \subseteq GL(2n)$. In each case the inclusion gives rise to a restriction map which takes any representation of the general linear group and views it as a module over the symplectic or orthogonal subgroup. If we pick one of these $G \subseteq GL(n)$ and an irreducible $GL(n)$ module $V(\lambda)$, its restriction $V(\lambda)|_G$ will in general be reducible, and the decomposition into symplectic or orthogonal irreducibles is independent of n as long as λ has at most n rows.

Taking characters translates this decomposition into the question of writing the Schur functions in the sp_λ or o_λ bases; the coefficients will be the multiplicities, so will certainly be nonnegative integers. The “branching rules” were known to Littlewood; they were written in the context of symmetric functions by Koike and Terada.

THEOREM 1.2. *The following summations are over all partitions μ .*

1. $s_\lambda = \sum_\mu \left(\sum_{\nu \in Y_{\square}} c_{\mu\nu}^\lambda \right) sp_\mu$, where $\nu \in Y_{\square}$ if the Young diagram of ν has only even-height columns, i.e. can be tiled by vertical dominos.
2. $s_\lambda = \sum_\mu \left(\sum_{\nu \in Y_{\square}} c_{\mu\nu}^\lambda \right) o_\mu$, where $\nu \in Y_{\square}$ if the Young diagram of ν has only even-length rows, i.e. can be tiled by horizontal dominos.

The names Y_{\square} and Y_{\square} are mnemonic, but they may be more familiar by other names. Viewed as sets of partitions, $\nu \in Y_{\square}$ means all parts of ν occur with even multiplicity, while $\nu \in Y_{\square}$ means ν has exclusively even parts. Viewed in terms of weights, $\nu \in Y_{\square}$ means ν is in the span of the even fundamental weights ω_{2i} , while $\nu \in Y_{\square}$ means the coefficient of ω_i in ν is even, for all i .

Now recall the remarkable fact from Theorem 1.1 that the sp_λ and o_λ bases have the same structure constants. The linear maps $sp_\lambda \mapsto o_\lambda$ and $o_\lambda \mapsto sp_\lambda$ are therefore ring isomorphisms. The representations we are related to the irreducible representations $V(\lambda)$ by these isomorphisms.

DEFINITION 1.3. We define two families of reducible representations, by giving their direct sum decomposition (with multiplicities) into irreducibles V_{Sp} or V_O of the symplectic or orthogonal groups, respectively:

$$W_{Sp}(\lambda) := \sum_\mu \left(\sum_{\nu \in Y_{\square}} c_{\mu\nu}^\lambda \right) V_{Sp}(\mu) \quad W_O(\lambda) := \sum_\mu \left(\sum_{\nu \in Y_{\square}} c_{\mu\nu}^\lambda \right) V_O(\mu)$$

One needs to compare Theorem 1.2 with Definition 1.3 carefully to distinguish the irreducible GL -module $V(\lambda)$ from these new $W_G(\lambda)$: the two differ typographically only by exchanging Y_{\square} with Y_{\square} . To avoid confusion, we place the four possibilities in one table to highlight their relations:

| | $\sum c_{\mu\nu}^\lambda$ | $V_{Sp}(\mu)$ | $V_O(\mu)$ |
|-----------------------|---------------------------|-------------------|----------------|
| $\nu \in Y_{\square}$ | | $V(\lambda)$ | $W_O(\lambda)$ |
| $\nu \in Y_{\square}$ | | $W_{Sp}(\lambda)$ | $V(\lambda)$ |

Note that the decomposition of $W_G(\lambda)$ is independent of n as long as λ has fewer than n rows. Therefore we can speak of the $W_G(\lambda)$ giving rise to stable limit characters, just as the $V(\lambda)$ give rise to the Schur functions as described in Section 1.1. In light of the similarity in their respective definitions, it is immediate that the stable limit characters of $W_{Sp}(\lambda)$ and $W_O(\lambda)$ are the images of s_λ under the ring isomorphisms $o_\mu \mapsto sp_\mu$ and $sp_\mu \mapsto o_\mu$, respectively. Because their characters are images of the Schur functions, we have a family of reducible representations of the symplectic or orthogonal groups whose tensor products decompose into a direct sum of family members according to the classical Littlewood–Richardson numbers:

$$(1.6) \quad W(\mu) \otimes W(\nu) \cong \sum_\lambda c_{\mu\nu}^\lambda W(\lambda), \quad \text{where } W = W_{Sp} \text{ or } W_O.$$

In [11] we showed that the $W(\lambda)$ are completely characterized by this property:

THEOREM 1.4. *Let $\{X(\lambda)\}$ be a family of representations of the symplectic or orthogonal groups G , indexed by all partitions λ and given in terms of their irreducible decompositions $X(\lambda) \cong V_G(\lambda) \oplus \sum_{\mu < \lambda} m_{\lambda\mu} V_G(\mu)$, for some nonnegative integers $m_{\lambda\mu}$. Suppose their tensor products decompose into direct sums as $X(\mu) \otimes X(\nu) \cong \sum_\lambda c_{\mu\nu}^\lambda X(\lambda)$. Then there are only two possibilities:*

- (a) $\{X(\lambda)\}$ is the family of GL -irreducible representations $\{V(\lambda)\}$; or
- (b) $\{X(\lambda)\}$ is the family $\{W_G(\lambda)\}$ defined above.

We have defined $W_G(\lambda)$ to be a representation of G , but for most of the remainder of this paper we will be primarily concerned with Lie algebras (and their loop algebras and quantum deformations). We therefore allow $W_G(\lambda)$ to denote a representation of the finite-dimensional Lie algebra \mathfrak{g} as well. Since we defined the representation in terms of its direct sum decomposition into irreducibles, nothing new is introduced by this convenience.

1.4. Combinatorics of \mathbf{W}_G . We will need the ability to compute $W_G(\lambda)$ explicitly. The definition in terms of Littlewood–Richardson numbers can be restated in terms of the *skewing* operation $s_\nu^\perp : \Lambda \rightarrow \Lambda$, the adjoint to multiplication by s_ν . The skew Schur function $s_{\lambda/\nu}$ is defined as $s_\nu^\perp s_\lambda$. The skew Young diagram for λ/ν is represented by the Young diagram for λ with the boxes for the Young diagram of ν removed from the upper-left corner; this notation relies on the fact that $s_{\lambda/\nu} = 0$ unless $\nu \subseteq \lambda$, where \subseteq denotes containment of Young diagrams. We mention that $s_{\lambda/\nu}$ also has a Jacobi–Trudi expansion:

$$s_{\lambda/\nu} = \det(h_{\lambda_i - \nu_j - i + j})_{i,j=1,\dots,r}.$$

The character of $W_O(\lambda)$ can now be described as the image of $\sum_{\nu \in Y_\blacksquare} s_{\lambda/\nu}$ under the linear map $s_\mu \mapsto o_\mu$, and similarly for $W_{Sp}(\lambda)$ with $\nu \in Y_\square$ and $s_\mu \mapsto sp_\mu$. Since we will focus on the decomposition into irreducibles, we just need to expand $\sum_{\nu \in Y_\blacksquare} s_{\lambda/\nu}$ (or $\nu \in Y_\square$) in the Schur basis. To calculate this character for a given λ , we will give a combinatorial algorithm to expand each $s_{\lambda/\nu}$ as a sum of Schur functions. Our presentation is heavily abridged, to say the least; we refer the reader to Chapter 5 of [4] for a rational and justified development.

DEFINITION 1.5. (a) A *semi-standard Young tableau* of shape λ/ν is a filling of the boxes of the skew Young diagram of λ/ν with nonnegative integers such that the entries *strictly increase* reading down any column and *weakly increase* reading across any row.

(b) The *reverse row word* of a tableau T is the sequence of integer entries of T as you read each row from right to left, beginning with the top row and ending with the bottom.

(c) A sequence of integers is a *ballot sequence* if the number of occurrences of $i+1$ in the first k terms is never greater than the number of occurrences of i , for every integer i and k .

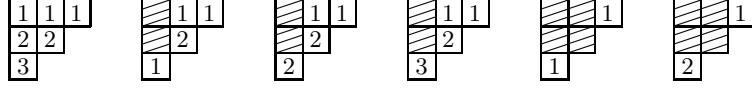
(d) The *content* of a tableau T is $\langle n_1(T), n_2(T), \dots \rangle$, where $n_i(T)$ is the number of occurrences of i in T .

PROPOSITION 1.6.

$$s_{\lambda/\nu} = \sum_T s_{\text{content}(T)},$$

where T ranges over all semi-standard Young tableaux of shape λ/ν such that the reverse row word of T is a ballot sequence. Since the reverse row word of T is a ballot sequence, $\text{content}(T)$ is a partition.

To illustrate, let us compute $W_O(\lambda)$ for $\lambda = \omega_1 + \omega_2 + \omega_3$. The only partitions $\nu \in Y_{\square}$ contained in λ are the empty partition, $\langle 1, 1 \rangle$, and $\langle 2, 2 \rangle$. The six semi-standard Young tableaux T whose reverse row words are ballot sequences are:



To convert content(T) into a weight, recall that the coefficient of ω_i is the number of occurrences of i minus the number of occurrences of $i+1$. So $W_O(\omega_1 + \omega_2 + \omega_3)$ decomposes as a sum of six components, respectively:

$$W_O(\omega_1 + \omega_2 + \omega_3) \oplus W_O(2\omega_1 + \omega_2) \oplus W_O(2\omega_2) \oplus W_O(\omega_1 + \omega_3) \oplus W_O(2\omega_1) \oplus W_O(\omega_2).$$

PROPOSITION 1.7. *Immediate consequences of the combinatorics of $W_G(\lambda)$:*

1. *If $V_G(\mu)$ appears in $W_G(\lambda)$ then $\mu \subseteq \lambda$, where \subseteq denotes containment of Young diagrams. In particular, if $\mu \neq \lambda$ then $|\mu| < |\lambda|$. As we observed at the end of Section 1.2, this is equivalent to saying that the root $\lambda - \mu$ is supported on the root α_n (and for $SO(2n)$ on α_{n-1} also).*
2. *The trivial representation $V_G(0) = \mathbf{C}$ appears in $W_O(\lambda)$ if and only if $\lambda \in Y_{\square}$, and in $W_{Sp}(\lambda)$ if and only if $\lambda \in Y_{\square\square}$. Moreover, if it does appear, it has multiplicity 1. This follows because $s_{\lambda/\nu}$ is homogeneous of degree $|\lambda| - |\nu|$, which by the previous point is zero if and only if $\nu = \lambda$ (when $s_{\lambda/\nu} = 1$).*

The first consequence will be a significant motivation for the conjecture in the next section, where another family of representations is uniquely characterized by the similar property that $\lambda - \mu$ cannot be contained in a sublattice corresponding to a type A subalgebra. We do not know of a way to see the second consequence from the representation theory side.

2. Loop algebras and their representations

From now on, \mathfrak{g} denotes a symplectic or orthogonal algebra of rank n and $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$ the corresponding loop algebra with the obvious Lie algebra structure. In this section, we construct a family of finite-dimensional indecomposable representations $W^{\text{aff}}(\lambda)$ of $L(\mathfrak{g})$ and conjecture that these modules are isomorphic as \mathfrak{g} -modules to the modules $W(\lambda)$ of the previous section.

2.1. Loop algebras. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and a set of simple roots, $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, we shall assume that the simple roots are numbered as in [7]. Thus in the case of $sp(2n)$ (resp. $so(2n+1)$), we assume that α_n is the long (resp. short) root, while in the case of $so(2n)$, we assume that $n-1$ and n are the spin nodes. In all cases, we assume in addition that

$$a_{i,i\pm 1} = -1, \quad 1 \leq i \leq n-2, \quad a_{ij} = 0, \quad j \neq i \pm 1.$$

Set $J = \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}\}$. In the case of $so(2n)$, we shall also need the subset $\overline{J} = \{\alpha_1, \dots, \alpha_{n-2}, \alpha_n\}$.

Let R^+ be the set of positive roots and let Q^+ (resp. P^+) be the integral root (resp. weight) lattice respectively. Let $\omega_1, \dots, \omega_n$, be a set of fundamental weights in P^+ . Given any subset J' of $\alpha_1, \alpha_2, \dots, \alpha_n$, let $R^+(J')$ be the subset of R^+ spanned by elements of J' . We define $Q^+(J')$ etc. in the obvious way.

For each $\alpha \in R^+$, fix nonzero elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ and $h_\alpha \in \mathfrak{h}$ satisfying,

$$[h_\alpha, x_\alpha^\pm] = \pm\alpha(h)x_\alpha^\pm, \quad [x_\alpha^+, x_\alpha^-] = h_\alpha.$$

If $\alpha, \beta \in R^+$ is such that $\alpha + \beta \in R^+$ or $\alpha - \beta \in R^+$ we shall assume that

$$[x_\alpha^\pm, x_\beta^\pm] = x_{\alpha+\beta}^\pm, \quad [x_\alpha^\pm, x_\beta^\mp] = x_{\alpha-\beta}^\pm.$$

Strictly speaking, the preceding statement is true only up to some nonzero scalar multiples, but for our purposes there is no loss in assuming, to simplify notation, that these scalars are all one.

We shall need the following subalgebras of \mathfrak{g} :

$$\begin{aligned} \mathfrak{k} &= \mathfrak{h} \oplus_{\alpha \in R^+(J)} \mathfrak{g}_\alpha, & \mathfrak{p} &= \mathfrak{k} \oplus \mathfrak{u}^+, \\ \mathfrak{u}^\pm &= \oplus_{\alpha \in R^+ \setminus R^+(J)} \mathfrak{g}_{\pm\alpha}, & \mathfrak{n}^\pm &= \oplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}. \end{aligned}$$

Then \mathfrak{u}^+ is an ideal in \mathfrak{p} and we have a homomorphism of Lie algebras $L(\mathfrak{p}) \rightarrow L(\mathfrak{k})$ with kernel $L(\mathfrak{u}^+)$. The algebra \mathfrak{k} is reductive, and hence given any $\lambda \in P^+$ there exists an irreducible \mathfrak{k} -module $V_{\mathfrak{k}}(\lambda)$ with highest weight λ and highest weight vector v_λ . This can clearly be regarded as a module for $L(\mathfrak{k})$ by composing with the evaluation homomorphism $ev : L(\mathfrak{k}) \rightarrow \mathfrak{k}$ that sends $x \otimes t^n \rightarrow x$ and hence also as a module for $L(\mathfrak{p})$.

2.2. Construction of \mathbf{W}^{aff} . Consider the induced module

$$\text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda) = \mathbf{U}(L(\mathfrak{g})) \otimes_{L(\mathfrak{p})} V_{\mathfrak{k}}(\lambda),$$

where for any Lie algebra \mathfrak{a} , we let $\mathbf{U}(\mathfrak{a})$ be the universal enveloping algebra of \mathfrak{a} . Since

$$(x_\alpha^+ \otimes t^n)(1 \otimes v_\lambda) = 0, \quad \forall \alpha \in R^+, n \in \mathbf{Z},$$

an elementary application of the Poincaré–Birkhoff–Witt theorem shows that,

$$\text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda) = \mathbf{U}(L(\mathfrak{n}^-))(1 \otimes v_\lambda) = \mathbf{U}(L(\mathfrak{u}^-)) \otimes V_{\mathfrak{k}}(\lambda).$$

In particular, the subspace

$$\left(\text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda) \right)_\lambda = \{v \in \text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda) : hv = \lambda(h)v \ \forall h \in \mathfrak{h}\}$$

has dimension one and hence $\text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda)$ has a unique irreducible quotient obtained as follows. Let $ev : L(\mathfrak{g}) \rightarrow \mathfrak{g}$ be the Lie algebra homomorphism obtained by mapping $t \mapsto 1$. This gives a $L(\mathfrak{g})$ -module structure on any \mathfrak{g} -module, in particular on the irreducible \mathfrak{g} -module $V(\lambda)$ with highest weight λ . It is easy to see that the $L(\mathfrak{g})$ -module $V(\lambda)$ is the unique irreducible quotient of $\text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda)$.

Assume from now on that $\lambda(h_{\alpha_n}) = 0$ (and $\lambda(h_{\alpha_{n-1}}) = 0$ in the case of $so(2n)$). It is easy to see in this case that the element $x_{\alpha_n}^- \otimes 1 \cdot (1 \otimes v_\lambda)$ (and $x_{\alpha_{n-1}}^- \otimes 1 \cdot (1 \otimes v_\lambda)$, in the case of $so(2n)$) generates a proper submodule of $\text{Ind}_{L(\mathfrak{p})}^{L(\mathfrak{g})} V_{\mathfrak{k}}(\lambda)$. Denote by $W^{\text{aff}}(\lambda)$ the corresponding $L(\mathfrak{g})$ -module quotient, and let $w_\lambda \in W^{\text{aff}}(\lambda)$ be the image of $1 \otimes v_\lambda$.

PROPOSITION 2.1.

(i) *As an $L(\mathfrak{g})$ -module, $W^{\text{aff}}(\lambda)$ is generated by w_λ with the following relations:*

$$x_\alpha^+ \otimes t^r \cdot w_\lambda = 0, \quad h \otimes t^r \cdot w_\lambda = \lambda(h)w_\lambda, \quad \alpha \in R^+, \quad h \in \mathfrak{h},$$

$$(x_{\alpha_i}^-)^{\lambda(h_i)+1} \cdot w_\lambda = 0, \quad x_\beta^- \otimes (t-1)t^r \cdot w_\lambda = 0, \quad \forall \beta \in R^+(J), \quad r \in \mathbf{Z}.$$

(ii) For all $\lambda \in P^+$, $\dim W^{\text{aff}}(\lambda) < \infty$.

PROOF. Part (i) is clear from the definition of $W^{\text{aff}}(\lambda)$. Part (ii) was proved in [2]. In the language of that paper, it is easy to see that the module $W^{\text{aff}}(\lambda)$ is a quotient of $W(\pi)$, where $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ is such that $\pi_i(u) = (1-u)^{\lambda(h_{\alpha_i})}$ for all $1 \leq i \leq n$. \square

In view of the preceding proposition, we can write

$$W^{\text{aff}}(\lambda) = \bigoplus_{\mu \in P^+} m_{\lambda, \mu} V(\mu).$$

Clearly, $m_{\lambda, \lambda} = 1$ and $m_{\lambda, \mu} = 0$ if $\lambda - \mu \notin Q^+$. We can now state our conjecture.

Conjecture 1. As \mathfrak{g} -modules, we have

$$W^{\text{aff}}(\lambda) \cong W(\lambda).$$

Here $W(\lambda)$ is the representation $W_O(\lambda)$ or $W_{Sp}(\lambda)$ of \mathfrak{g} defined in Section 1.3.

2.3. Type A sublattices. As a first step towards providing evidence for this conjecture we prove the following result which is analogous to Proposition 1.7.

For $\eta = \sum_i s_i \alpha_i \in Q^+$, set $\text{supp } \eta = \{\alpha_i : s_i \neq 0\}$ and let $J(\eta)$ be the minimal connected subset of the set of simple roots that contains $\text{supp } \eta$. Let $\mathfrak{g}(J_\eta)$ be the subalgebra generated by the elements $x_{\pm \alpha}$, $\alpha \in R^+(J_\eta)$. We say that η is of type A if the elements $x_{\alpha_i}^\pm$, $\alpha_i \in J(\eta)$ generate a subalgebra of \mathfrak{g} of type sl_r .

PROPOSITION 2.2. For all $0 \neq \eta \in Q^+$ of type A, we have

$$m_{\lambda, \lambda - \eta} = 0.$$

PROOF. Suppose first that $\alpha_n \notin \text{supp } \eta$. Then $\text{supp } \eta \subset J$ and hence

$$W^{\text{aff}}(\lambda)_{\lambda - \eta} = 1 \otimes V_{\mathfrak{k}}(\lambda)_{\lambda - \eta}.$$

Since $V_{\mathfrak{k}}(\lambda)$ is an irreducible \mathfrak{k} -module, it follows that $w = 1 \otimes v_\lambda$.

If $\alpha_n \in \text{supp } \eta$, then since η is of type A, one can conclude by a simple inspection that one of the following must hold.

Case 1. If \mathfrak{g} is of type $sp(2n)$ or $so(2n+1)$, then $\text{supp } \eta = \{\alpha_n\}$ and hence $\eta = s\alpha_n$ for some $s \geq 0$. By definition, we have

$$x_{\alpha_n}^- \otimes 1 \cdot w_\lambda = 0.$$

Applying $h_{\alpha_n} \otimes t^r$ to the above equation, we see again from the definition that

$$x_{\alpha_n}^- \otimes t^r \cdot w_\lambda = 0.$$

Hence $W^{\text{aff}}(\lambda)_{\lambda - \eta} = 0$ thus proving the proposition.

Case 2. Suppose that \mathfrak{g} is of type $so(2n)$. If $\alpha \in R^+(\text{supp } \eta) \setminus R^+(J)$, then it is easy to see that there exists $\beta \in R^+(J_\eta)$ such that $\alpha = \beta + \alpha_n$ or $\alpha = \beta + \alpha_{n-1}$ or $\alpha = \beta + \alpha_{n-1} + \alpha_n$. Writing $x_\alpha^- = [x_\beta^-, x_{\alpha_n}^-]$ etc. we see in all cases that

$$x_\alpha^- \otimes (t-1)t^r \cdot w_\lambda = 0, \quad \forall r \in \mathbf{Z}.$$

Hence $\mathbf{U}(L(\mathfrak{g}(J_\eta))).w_\lambda = \mathbf{U}(\mathfrak{g}(J_\eta)).w_\lambda$. Since $\mathbf{U}(\mathfrak{g}(J_\eta)).w_\lambda$ is an irreducible $\mathfrak{g}(J_\eta)$ -module, and $W^{\text{aff}}(\lambda)_{\lambda - \eta} \subset \mathbf{U}(\mathfrak{g}(J_\eta)).w_\lambda$, the proposition follows. \square

2.4. Roots to consider. In this section we examine the structure of $L(\mathfrak{g})$ to determine which $V(\mu)$ can possibly appear as summands in $W^{\text{aff}}(\lambda)$, or equivalently, which $m_{\lambda,\mu}$ might be nonzero. The main result is Proposition 2.6, which says that $\lambda - \mu$ must lie in the \mathbf{Z}_+ -span of a special subset of the positive roots.

Although the module $W^{\text{aff}}(\lambda)$ is not an evaluation module, the next result shows that it is in fact a module for the quotient of $L(\mathfrak{g})$ by the ideal $\mathfrak{g} \otimes (t-1)^2 \mathbf{C}[t, t^{-1}]$.

PROPOSITION 2.3.

(i) Let $\alpha \in R^+$ be such that $\alpha - \alpha_n \in R^+(J)$ (or $\alpha - \alpha_{n-1} \in R^+(\bar{J})$ when $\mathfrak{g} = \text{so}(2n)$). Then for all $r \in \mathbf{Z}$ with $|r| \geq 1$,

$$(x_\alpha^- \otimes (t-1)^r).w_\lambda = 0.$$

(ii) For all $\alpha \in R^+$, and $r \in \mathbf{Z}$ with $|r| \geq 2$, we have,

$$(x_\alpha^- \otimes (t-1)^r).w_\lambda = 0.$$

PROOF. To prove (i), recall that

$$x_\alpha^- \otimes (t-1)^r = [x_{\alpha_n}^- \otimes 1, x_{\alpha-\alpha_n}^- \otimes (t-1)^r].$$

Part (i) now follows from the defining relations in $W^{\text{aff}}(\lambda)$.

It suffices to prove (ii) in the case of θ , the highest root. A simple case by case inspection shows that we can write $\theta = \alpha + \beta$ where either

- (a) $\alpha \in R^+(J)$, $\beta \in R^+$, $\beta - \alpha_n \in R^+(J)$, or
- (b) both $\alpha, \beta \in R^+$ and $\alpha - \alpha_n, \beta - \alpha_n \in R^+(J)$,
- (c) both $\alpha, \beta \in R^+$ and $\alpha - \alpha_n, \beta - \alpha_{n-1} \in R^+(\bar{J})$.

The result now follows from part (i). □

In view of the preceding proposition, we shall be interested in the following subset of R^+ ,

$$\{\beta \in R^+ : \beta, \beta - \alpha_n \notin R^+ \setminus R^+(J)\}.$$

(or, if $\mathfrak{g} = \text{so}(2n)$, the set

$$\{\beta \in R^+ : \beta, \beta - \alpha_n, \beta - \alpha_{n-1} \notin R^+ \setminus (R^+(J) \cup R^+(\bar{J}))\}.$$

Let N be the cardinality of this set.

We write the sets out explicitly for the reader's convenience. The sets in question can be written as the collection of roots $\beta_{k,l}$, defined as follows.

$\mathfrak{g} = \text{sp}(2n)$. For $1 \leq k < l \leq n-1$, set

$$\beta_{k,l} = \alpha_k + \alpha_{k+1} + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n-1} + \alpha_n,$$

and set $\beta_{0,l} = 2\alpha_1 + \cdots + 2\alpha_{n_1} + \alpha_n$.

$\mathfrak{g} = \text{so}(2n+1)$. For $1 \leq k < l \leq n-1$, set

$$\beta_{k,l} = \alpha_k + \alpha_{k+1} + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_n.$$

$\mathfrak{g} = \text{so}(2n+2)$. For $1 \leq k < l \leq n-1$, set

$$\beta_{k,l} = \alpha_k + \alpha_{k+1} + \cdots + \alpha_{l-1} + 2\alpha_l + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}.$$

We order the roots $\beta_{k,l}$ in the lexicographic order,

$$\beta_{k,l} < \beta_{k',l'} \text{ if } k < k', \text{ or } k = k', l < l',$$

and let $\beta_1, \beta_2, \dots, \beta_N$ be an enumeration of these roots.

Given $\mathbf{r} \in \mathbf{Z}^N$, set

$$x_{\mathbf{r}}^- = (x_{\beta_1}^- \otimes (t-1))^{r_1} (x_{\beta_2}^- \otimes (t-1))^{r_2} \cdots (x_{\beta_N}^- \otimes (t-1))^{r_N}.$$

Let \leq be the lexicographic order on \mathbf{Z}^N . The following corollary to Proposition 2.3 is now immediate, by a simple application of the PBW theorem.

COROLLARY 2.4. *We have,*

$$W^{\text{aff}}(\lambda) = \sum_{\mathbf{r} \in \mathbf{Z}_+^N} \mathbf{U}(\mathbf{n}^-) x_{\mathbf{r}}^- \cdot w_{\lambda}$$

□

We can now give a necessary condition for $m_{\lambda, \mu}$ to be nonzero. We begin with the following Lemma which is easily checked.

LEMMA 2.5. (i) *For all $1 \leq r, s \leq N$ we have $\beta_r + \beta_s \notin R^+$.*
(ii) *For any $i = 1, \dots, n$ and for all $1 \leq r, s \leq N$, we have $\beta_r + \beta_s - \alpha_i \notin R^+$.*
(iii) *Suppose that $i = 1, \dots, n$ and $1 \leq r \leq N$ are such that $\beta_r - \alpha_i \in R^+$. Then, either $\beta_r - \alpha_i = \beta_s$ for some $s \geq r$, or $\beta_r = \beta_{k, n-1}$, $i = n-1$ for some k .*

□

PROPOSITION 2.6. *Assume that $m_{\lambda, \mu} \neq 0$. Then*

$$\mu = \lambda - \sum_{j=1}^N s_j \beta_j,$$

for some nonnegative integers s_1, s_2, \dots, s_N .

PROOF. Let W_1 be a \mathfrak{g} -module complement to $V(\lambda)$ so that we have

$$W^{\text{aff}}(\lambda) = V(\lambda) \oplus W_1,$$

as \mathfrak{g} -modules. If $W_1 \neq 0$, choose \mathbf{r}_1 minimal so that the projection $w_{\mathbf{r}_1}$ of $x_{\mathbf{r}_1}^- \cdot w_{\lambda}$ onto W_1 is nonzero. Using Lemma 2.5, we see that

$$x_{\alpha_i}^+ \cdot x_{\mathbf{r}_1} \cdot w_{\lambda} \in \sum_{\mathbf{s} < \mathbf{r}_1} \mathbf{U}(\mathbf{n}^-) \cdot x_{\mathbf{s}}^- \cdot w_{\lambda}.$$

The minimality of \mathbf{r}_1 now implies that

$$x_{\alpha_i}^+ \cdot x_{\mathbf{r}_1} \cdot w_{\lambda} \in V(\lambda),$$

which implies that

$$x_{\alpha_i}^+ \cdot w_{\mathbf{r}_1} \cdot w_{\lambda} = 0.$$

Hence, $w_{\mathbf{r}_1}$ generates an irreducible \mathfrak{g} -module. Let $W_2 \subset W_1$ be the \mathfrak{g} -module complement to it. Repeating the argument, we see that there exist a finite set $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ such that

$$W^{\text{aff}}(\lambda) = \bigoplus V(\mu_{\mathbf{r}_j}),$$

where $\mu_{\mathbf{r}_j}$ is the weight of the element $w_{\mathbf{r}_j}$. Clearly, each $\mu_{\mathbf{r}_j}$ has weight of the form $\lambda - \sum_j s_j \beta_j$ and the proposition follows. □

2.5. Stable limit property. We next prove an interesting consequence of the preceding proposition. It is the analogue of the statement following Definition 1.3 that the modules $W(\lambda)$ have the same decomposition for all sufficiently large n . We prove this for the orthogonal algebras; the case of symplectic algebras is similar and simpler.

For this proposition only, we denote by \mathfrak{g}_{2n} the Lie algebra $so(2n)$ and by \mathfrak{g}_{2n+1} the Lie algebra $so(2n+1)$. We denote the corresponding lattice Q^+ by Q_n^+ etc., and similarly denote $W^{\text{aff}}(\lambda)$ by $W_n^{\text{aff}}(\lambda)$, and the multiplicities $m_{\lambda,\mu}$ by $m_{\lambda,\mu,n}$.

We have an embedding of $so(2n-1) \rightarrow so(2n)$, given as follows,

$$x_{\alpha_i}^\pm \mapsto x_{\alpha_i}^\pm, \quad 1 \leq i \leq n-2, \quad x_{\alpha_{n-1}}^\pm \mapsto x_{\alpha_n} + x_{\alpha_{n-1}}.$$

In other words $so(2n-1)$ is the subalgebra of fixed points of the automorphisms of $so(2n)$ defined by interchanging the spin nodes of the Dynkin diagram.. Under this embedding, the root vectors

$$x_{\beta_{k,l}}^- \mapsto x_{\beta_{k,l}}^\pm, \quad 1 \leq k < l \leq n-1.$$

The restriction map $\mathfrak{h}_{2n}^* \rightarrow \mathfrak{h}_{2n-1}^*$ induces an isomorphism between the subspace of P_{2n}^+ spanned by ω_i , $1 \leq i \leq n-2$ and the subspace of P_{2n-1}^+ spanned by ω_i , $1 \leq i \leq n-2$.

We also define an embedding of $so(2n) \rightarrow so(2n+1)$. This is given by the assignment,

$$\begin{aligned} x_{\alpha_i}^\pm &\mapsto x_{\alpha_i}^\pm, \quad 1 \leq i \leq n-2, \\ x_{\alpha_{n-1}}^\pm &\mapsto x_{\alpha_{n-1}}^\pm, \quad x_{\alpha_n}^\pm \mapsto x_{\alpha_{n-1}+2\alpha_n}^\pm. \end{aligned}$$

Again notice that this embedding maps $\beta_{k,l}$ to $\beta_{k,l}$ for all $l \leq n-2$ and ω_i to ω_i for $i \leq n-2$.

Both embeddings naturally extend to maps of the corresponding loop algebras.

THEOREM 2.7. *Let $\lambda \in P_n^+$ and assume that $\lambda(h_n) = \lambda(h_{n-1}) = 0$. There exists $r(n) \geq n$ such that for all $s, s' \geq r(n)$, we have*

$$m_{\lambda,\mu,s} = m_{\lambda,\mu,s'}.$$

The theorem is clearly a consequence of the following proposition.

PROPOSITION 2.8. *Let $\lambda \in P_n^+$. Then,*

$$m_{\lambda,\mu,n} \geq m_{\lambda,\mu,n+1}.$$

PROOF. Assume first that $n = 2m+1$ and let N be the number of roots of the form $\beta_{i,k}$ for $so(2m+1)$. Notice that this is exactly the same number of such roots for $so(2m+2)$. By Proposition 2.4, we have

$$W_{2m+2}^{\text{aff}}(\lambda) = \sum_{\mathbf{r} \in \mathbf{Z}^N} \mathbf{U}(\mathfrak{n}_{2m+2}^-) x_{\mathbf{r}}^- \cdot w_\lambda.$$

The elements $x_{\mathbf{r}}^-$ are in the image of the embedding of $so(2m+1) \rightarrow so(2m+2)$. Hence, setting, $W_{2m+1} = \mathbf{U}(\mathfrak{g}_{2m+1}) \cdot w_\lambda$ we see that

$$W_{2m+1} = \sum_{\mathbf{r} \in \mathbf{Z}^N} \mathbf{U}(\mathfrak{n}_{2m+1}^-) \cdot w_\lambda.$$

It is now clear, that the elements w_s defined in the proof of Proposition 2.6 can be chosen to be in W_{2m+1} . It is clear from the defining relations of $W_{2m+1}^{\text{aff}}(\lambda)$ that

there exists a surjective map $W_{2m+1}^{\text{aff}}(\lambda) \rightarrow W_{2m+1}$ of \mathfrak{g}_{2m+1} -modules. This clearly implies that $m_{\mu,2m+2} \leq m_{\mu,2m+1}$.

To prove that $m_{\mu,2m+1} \leq m_{\mu,2m}$, we use the embedding of $so(2m) \rightarrow so(2m+1)$. The proof is similar, we just need to show that the elements $x_{\mathbf{r}}^-$ that span $W_{2m+1}^{\text{aff}}(\lambda)$ are actually in $\mathbf{U}(\mathfrak{n}_{2m})$. The only difficulty is with the roots $\beta_{k,n-1}$ and $\beta_{k,n}$. Now, $\beta = \beta_{k,2n-1} - \alpha_{2m-1} - \alpha_n \in R^+$ is such that $x_{\beta}^- \otimes (t-1).w_{\lambda} = 0$. Since $\lambda(h_{n-1}) = \lambda(h_n) = 0$, it follows that $x_{\alpha_{n-1}+\alpha_n}^- = 0$. This now implies that $x_{\beta_{k,n-1}}^- \otimes (t-1).w_{\lambda} = 0$. The case of $\beta_{k,n}$ is dealt with similarly. This completes the proof of the proposition. \square

3. Motivations and special cases of the conjecture

We have restricted ourselves to the case of the enveloping algebras of loop algebras to simplify matters and to avoid excessive notation. However, the motivation for Conjecture 1 comes from connections which we now explain, with the irreducible finite-dimensional representations of quantum affine algebras.

3.1. Background. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and the related Yangian $Y(\mathfrak{g})$ were introduced by Drinfeld and Jimbo as tools for studying solutions to the quantum Yang–Baxter equation. Finite dimensional representations of either Hopf algebra give rise to solutions (called R -matrices), which can, in turn, be used to construct the transfer matrices of integrable dynamical systems. The Bethe Ansatz is a technique for calculating eigenvalues of such transfer matrices, as the solutions to a set of algebraic equations.

The algebras above have subalgebras $U_q(\mathfrak{g}) \hookrightarrow U_q(\hat{\mathfrak{g}})$ and $\mathfrak{g} \hookrightarrow Y(\mathfrak{g})$, and the eigenspaces of the transfer matrix decompose a finite-dimensional representation of the larger algebra into subspaces stabilized by the smaller one. If each eigenspace were a single irreducible representation, then the eigenvalues of the transfer matrix would completely describe the decomposition of a representation of $U_q(\hat{\mathfrak{g}})$ or $Y(\mathfrak{g})$ upon restriction to $U_q(\mathfrak{g})$ or \mathfrak{g} , respectively. If, in addition, the Bethe Ansatz finds all eigenvalues, then solving the Bethe equations would yield the complete desired decomposition.

This was the approach employed by Kirillov and Reshetikhin [8]. They addressed the problem of decomposing an irreducible $Y(\mathfrak{g})$ module according to the action of the embedded copy of \mathfrak{g} by conjecturing that the Bethe Ansatz detected all the pieces in the decomposition. The result was a so-called ‘fermionic formula’ for the number of times each \mathfrak{g} module would appear in the decomposition. Their attention was restricted to a particular class of finite-dimensional representations in which the Bethe eigenvectors are especially well-behaved.

The results in this paper are all from the point of view of the embedding $U_q(\mathfrak{g}) \hookrightarrow U_q(\hat{\mathfrak{g}})$. It has long been a folk theorem that the decompositions in this case were identical to those in the $\mathfrak{g} \hookrightarrow Y(\mathfrak{g})$ case. A proof was recently given for simply-laced \mathfrak{g} by Varagnolo [14]. Further, we know by results of Lusztig [12] that the representation theory of $\mathbf{U}_q(\mathfrak{g})$ over $\mathbf{C}(q)$ is the ‘same’ as the representation theory of $\mathbf{U}(\mathfrak{g})$. Hence we are justified in talking about the Kirillov–Reshetikhin conjecture on Yangians as if it applies to the representations we studied in Section 2.

Conjecture of Kirillov and Reshetikhin. For each $m \in \mathbf{Z}^+$ and $\ell = 1, \dots, n$, there exists an irreducible representation $V_q(m\omega_{\ell})$ of $\mathbf{U}_q(\hat{\mathfrak{g}})$ whose highest

weight when viewed as a representation of $\mathbf{U}_q(\mathfrak{g})$ is $m\omega_\ell$. Further, the decomposition of the tensor product of N such representations as $\mathbf{U}_q(\mathfrak{g})$ -modules is given by

$$\bigotimes_{a=1}^N (V_q(m_a\omega_{\ell_a})|_{\mathfrak{g}}) \simeq \sum_{\lambda} n_{\lambda} V(\lambda)$$

where the sum runs over all weights λ less than $\sum m_a\omega_{\ell_a}$, the highest weight of the tensor product. The nonnegative integer n_{λ} is the multiplicity with which the irreducible \mathfrak{g} -module $V(\lambda)$ occurs. Write $\lambda = \sum m_a\omega_{\ell_a} - \sum n_i\alpha_i$. Then

$$n_{\lambda} = \sum_{\text{partitions}} \prod_{n \geq 1} \prod_{k=1}^r \binom{P_n^{(k)}(\nu) + \nu_n^{(k)}}{\nu_n^{(k)}}$$

The sum is taken over all ways of choosing partitions $\nu^{(1)}, \dots, \nu^{(r)}$ such that $\nu^{(i)}$ is a partition of n_i which has $\nu_n^{(i)}$ parts of size n (so $n_i = \sum_{n \geq 1} n\nu_n^{(i)}$). The function P is defined by

$$\begin{aligned} P_n^{(k)}(\nu) = & \sum_{a=1}^N \min(n, m_a) \delta_{k, \ell_a} - 2 \sum_{h \geq 1} \min(n, h) \nu_h^{(k)} + \\ & + \sum_{j \neq k}^r \sum_{h \geq 1} \min(-c_{k,j}n, -c_{j,k}h) \nu_h^{(j)} \end{aligned}$$

where $C = (c_{i,j})$ is the Cartan matrix of \mathfrak{g} , and $\binom{a}{b} = 0$ whenever $a < b$.

The formula describing the n_{λ} is called the fermionic formula. The connection with representation theory was made by Kirillov and Reshetikhin, who proved the conjecture in the case of sl_n .

The fermionic formula is somewhat difficult to work with directly. However, when \mathfrak{g} is classical and we ignore the tensor product (taking $N = 1$), there is a simple combinatorial description of what the fermionic formula predicts for $V_q(m\omega_\ell)$; for a derivation of the combinatorics from the fermionic formula see [10]. In particular, as long as the weight ω_ℓ lies in the type A part of the Dynkin diagram ($\ell < n$ and $\ell < n - 1$ for $so(2n)$), the Kirillov–Reshetikhin decompositions are a special case of the ones we defined in Section 1: the $\mathbf{U}_q(\mathfrak{g})$ module structure of $V_q(m\omega_\ell)$ is the same as the \mathfrak{g} -module structure of $W_O(m\omega_\ell)$ when \mathfrak{g} is orthogonal, and is the same as $W_{Sp}(m\omega_\ell)$ when \mathfrak{g} is symplectic. Thus it becomes natural to make the following conjecture.

Conjecture 2. There exists an irreducible representation $V_q^{\text{aff}}(\lambda)$ of the quantum affine algebra whose $\mathbf{U}_q(\mathfrak{g})$ -module decomposition is $W_G(\lambda)$.

3.2. Minimal affinizations. It is known [1] that the finite-dimensional irreducible representations of quantum affine algebras $\mathbf{U}_q(\mathfrak{g})$ are indexed by the n -tuples (π_1, \dots, π_n) of polynomials with constant term 1. In [3], we showed that the module $V_q(m\omega_\ell)$ conjectured by Kirillov and Reshetikhin is given by the n -tuple

$$\pi_j = 1, \quad j \neq m, \quad \pi_m = (1 - q^{-\ell+1}u)(1 - q^{-\ell+3}u) \cdots (1 - q^{\ell-1}u).$$

These modules are the so called minimal affinizations of $m\omega_\ell$, see [3]. However, minimal affinizations $V_q^{\text{aff}}(\lambda)$ are known to exist more generally for any dominant integral weight λ [1], and as long as λ is not supported on the spin nodes, as in

the previous section, they are unique (up to $\mathbf{U}_q(\mathfrak{g})$ -module isomorphism). In fact, these modules are determined by the requirement that

$$m_{\lambda, \lambda - \eta} = 0 \quad \text{for all } \eta \neq 0 \text{ not supported on the spin nodes,}$$

a condition satisfied by the modules in Sections 1 and 2 of this paper.

It can be shown as in [3] that on specializing this representation, by putting $q = 1$, we get a quotient of the module $W^{\text{aff}}(\lambda)$. Thus, to prove our Conjecture 2 generalizing the Kirillov–Reshetikhin decompositions, it suffices to prove Conjecture 1 along with the statement that the specialized module is isomorphic to $W^{\text{aff}}(\lambda)$.

In the rest of this section, we restrict ourselves to the case of $so(2n)$ and explicitly calculate $W(\lambda) = W_O(\lambda)$ for several families of λ . We also show that the modules $W^{\text{aff}}(\lambda)$ are isomorphic to a submodule of $W(\lambda)$ in these cases. To prove that it is isomorphic to $W(\lambda)$ requires arguments in the quantum algebra, similar to the ones in [3] and we do not give details of that here.

3.3. Computation of examples. Our computations use the technique described in Section 1.4. We begin with the simplest case.

EXAMPLE 3.1 (Rectangles). When λ is a multiple of a fundamental weight $m\omega_\ell$, its Young diagram is a rectangle with ℓ rows and m columns. The reader can readily verify that for each $\nu \in Y_\square$, there is a unique semi-standard tableau for λ/ν whose reverse row word is a ballot sequence:

| | | | |
|---------|---------|---------|---------|
| | | | 1 ··· 1 |
| | | | 2 ··· 2 |
| | 1 ··· 1 | 3 ··· 3 | |
| | 2 ··· 2 | 4 ··· 4 | |
| 1 ··· 1 | 3 ··· 3 | 5 ··· 5 | |

The Young diagram of its content μ is that of λ/ν rotated 180° . Thus $W(m\omega_\ell)$ is the sum of $V(\mu)$ over all dominant weights μ which can be obtained from $\lambda = m\omega_\ell$ by repeatedly subtracting some fundamental weight ω_i and adding ω_{i-2} instead: each replacement of ω_i by ω_{i-2} corresponds to a vertical domino in ν removing two boxes from a column of height i .

The proof that $W^{\text{aff}}(\lambda) \cong W_O(\lambda)$ was given in [3].

EXAMPLE 3.2. Now take $\lambda = a\omega_1 + b\omega_2 + c\omega_3$; for D_5 this is the generic weight not supported on the spin nodes. This example is simple because the only $\nu \in Y_\square$ that fit inside λ consist of two rows of equal length. The typical tableau this time is of the following form:

| | | | | | |
|--|---------|---------|---------|---------|---------|
| | c | | b | a | |
| | | | | | |
| | | | 1 ··· 1 | 1 ··· 1 | 1 ··· 1 |
| | | | 2 ··· 2 | 2 ··· 2 | |
| | 1 ··· 1 | 2 ··· 2 | 3 ··· 3 | 3 ··· 3 | |
| | t | s | r | | |

Here ν has two rows of length $r + s + t$, which is depicted here as being less than c , but can also lie between c and $b + c$. In either case, by definition

$$s + t \leq c.$$

Two additional restrictions on the parameters are imposed by the requirement that the reverse row word be a ballot sequence:

$$\begin{aligned} s &\leq a \\ r &\leq b \end{aligned}$$

Requiring $r \leq b$ ensures that the number of 2s never exceeds the number of 3s; this is still the correct bound no matter how $r + s + t$ compares with c .

Converting the content of the tableau into a weight μ , we find that

$$\mu = (a - s + t)\omega_1 + (b - r + s)\omega_2 + (c - s - t)\omega_3.$$

Thus the decomposition for $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ is

$$W(\lambda) = \sum_{\substack{s \leq a, r \leq b, \\ s+t \leq c}} V(\lambda - r(\omega_2) - s(\omega_3 - \omega_2 + \omega_1) - t(\omega_3 - \omega_1))$$

We have rewritten μ to highlight the fact that each of r , s and t count the number of times some weight is subtracted from λ .

We now turn to the $L(\mathfrak{g})$ -modules $W^{\text{aff}}(\lambda)$ for D_5 . In this case we take $J = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The roots $\beta_{k,l}$ of Section 2 can also be written as

$$\beta_{1,2} = \omega_2, \quad \beta_{1,3} = \omega_1 - \omega_2 + \omega_3, \quad \beta_{2,3} = \omega_3 - \omega_1.$$

Thus, Proposition 2.6 gives us that

$$W^{\text{aff}}(\lambda) = \bigoplus m_{\lambda,\mu} V(\mu),$$

where

$$\mu \in \{\lambda - r_1\omega_2 - r_2(\omega_3 - \omega_2 + \omega_1) - r_3(\omega_3 - \omega_1) : r_1, r_2, r_3 \geq 0\}.$$

This immediately gives

$$m_{\lambda,\mu} \neq 0 \implies c \geq r_2 + r_3.$$

It remains to prove that $r_1 \leq b$ and $r_2 \leq a$ if $m_{\lambda,\mu} \neq 0$. To do this it is obviously enough to establish the following lemma.

LEMMA 3.3. *Let $\mathbf{r} \in \mathbf{Z}^3$. Then,*

$$x_{\mathbf{r}}^- \cdot w_{\lambda} \in \sum_{\mathbf{s} \leq (b, a, r_3)} \mathbf{U}(\mathfrak{g}) \cdot x_{\mathbf{s}}^- \cdot w_{\lambda}$$

PROOF. Set $N = \sum_{\mathbf{s} \leq (b, r_2, r_3)} \mathbf{U}(\mathfrak{g}) \cdot x_{\mathbf{s}}^- \cdot w_{\lambda}$. Observe that,

$$(x_{\alpha_2}^-)^a \cdot x_{\mathbf{r}}^- = x_{\mathbf{r}}^- x_{\alpha_2}^- + x_{r_1+1, r_2-1, r-3+1}^-.$$

If $r_1+1 \leq b$ this implies that $x_{\mathbf{r}} x_{\alpha_2}^- \cdot w_{\lambda} \in N$. Repeating this, we see that $x_{\mathbf{r}} (x_{\alpha_2}^-)^l \in N$ if $r_1 + l \leq b$. Taking $l = b$, $r_1 = 0$, we get $x_{0, r_2, r_3} (x_{\alpha_2}^-)^b \cdot w_{\lambda} \in N$. Since $(x_{\alpha_2}^-)^{b+1} \cdot w_{\lambda} = 0$, we can apply $x_{\alpha_2}^-$ to $x_{(0, r_2, r_3)} (x_{\alpha_2}^-)^b \cdot w_{\lambda}$ to find that

$$x_{(1, r_2-1, r_3)} \cdot w_{\lambda} \in N$$

for all r_2 . But $x_{(1, r_2, r_3)} (x_{\alpha_2}^-)^a - 1 \cdot w_{\lambda} \in N$, hence we get $x_{(2, r_2-1, r_3)} \cdot (x_{\alpha_2}^-)^{a-1} \cdot w_{\lambda} \in N$. Continuing, we get $x_{(r_1, r_2, r_3)} \cdot w_{\lambda} \in N$.

A similar argument shows how to reduce to $r_2 \leq a$. We omit the details. \square

EXAMPLE 3.4. All the decompositions $W(\lambda)$ calculated so far have been multiplicity free. This is not the case in general; for completeness we include the minimal counterexample, $\lambda = \omega_2 + \omega_4$. This time there are five $\nu \in Y_{\emptyset}$ that fit in λ , giving rise to seven semi-standard Young tableaux with ballot sequences for reverse row words:

| | | | | | | |
|--|--|--|--|--|--|--|
| | | | | | | |
|--|--|--|--|--|--|--|

Two tableaux have content $\mu = \omega_2$, and therefore $V(\omega_2)$ occurs with multiplicity two in $W(\omega_2 + \omega_4)$. We leave it to the enterprising reader to check that in general,

$$W(a\omega_2 + b\omega_4) = \sum_{\mu} m_{\mu} V(\mu),$$

where the sum is over all $\mu = c_1\omega_1 + c_2\omega_2 + c_3\omega_3 + c_4\omega_4$ such that $\mu \subseteq a\omega_2 + b\omega_4$ and $c_1 = c_3 \leq a$, and with multiplicities m_{μ} given by

$$m_{\mu} = 1 + \min(c_2, a - c_3, b - c_3 - c_4, a + b - c_1 - c_2 - c_3 - c_4).$$

The same techniques used in the previous example can be used here. One first identifies the minimal subset of roots $\beta_{k,l}$ such that $x_{\beta_{k,l}}^- \otimes (t-1).w_{\lambda} \neq 0$. A simple counting argument then gives us the maximal possible value for $m_{\lambda,\mu}$. Then one can prove analogues of Lemma 3.3 in exactly the same way, to give an upper bound on $m_{\lambda,\mu}$. When $a = b = 1$ this upper bound is precisely the multiplicity m_{μ} given above. In particular, when $\mu = \omega_2$, the bound $m_{\lambda,\mu} \leq 2$ arises because the difference $\lambda - \mu$ can be written either as $\beta_{1,2} + \beta_{3,4}$ or as $\beta_{1,3} + \beta_{2,4}$.

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